

On the distance from a matrix polynomial to matrix polynomials with k prescribed distinct eigenvalues

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Abstract

Consider an $n \times n$ matrix polynomial $P(\lambda)$ and a set Σ consisting of $k \leq n$ distinct complex numbers. In this paper, a (weighted) spectral norm distance from $P(\lambda)$ to the matrix polynomials whose spectra include the specified set Σ , is defined and studied. An upper and a lower bounds for this distance are obtained, and an optimal perturbation of $P(\lambda)$ associated to the upper bound is constructed. Numerical examples are given to illustrate the efficiency of the proposed bounds.

Keywords: Matrix polynomial, Eigenvalue, Perturbation, Singular value.

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1 Introduction

Let A be an $n \times n$ complex matrix and let \mathcal{M} be the set of all $n \times n$ complex matrices that have $\mu \in \mathbb{C}$ as a multiple eigenvalue. Malyshev [13] obtained the following singular value optimization characterization for the spectral norm distance from A to \mathcal{M} :

$$\min_{B \in \mathcal{M}} \|A - B\|_2 = \max_{\gamma \geq 0} s_{2n-1} \left(\begin{bmatrix} A - \mu I & \gamma I_n \\ 0 & A - \mu I \end{bmatrix} \right),$$

where $\|\cdot\|_2$ denotes the spectral matrix norm subordinate to the euclidean vector norm, and s_i is the i th singular value of the corresponding matrix ordered in a nonincreasing order. Malyshev's work can be considered as a solution to Wilkinson's problem, that is, the computation of the distance from a matrix $A \in \mathbb{C}^{n \times n}$ with all its eigenvalues simple to the $n \times n$ matrices that have multiple eigenvalues. This distance was introduced by Wilkinson in [24], and some bounds for it were computed by Ruhe [18], Wilkinson [20–23] and Demmel

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[2]. A spectral norm distance from A to matrices that have a prescribed eigenvalue of algebraic multiplicity 3, or any prescribed algebraic multiplicity, were obtained by Ikramov and Nazri [6] and Mengi [15], respectively. Moreover, Lippert [12] and Gracia [5] studied a spectral norm distance from A to the matrices with two prescribed eigenvalues, and obtained a nearest matrix to A having these two eigenvalues.

In 2008, Papathanasiou and Psarrakos [16] generalized Malyshev's results for the case of matrix polynomials, introducing a (weighted) spectral norm distance from an $n \times n$ matrix polynomial $P(\lambda)$ to the matrix polynomials that have a prescribed $\mu \in \mathbb{C}$ as a multiple eigenvalue, and obtaining an upper and a lower bounds for this distance. Lately, motivated by Mengi's results in [15], Psarrakos [17] introduced the matrix polynomials

$$F_k [P(\lambda); \gamma] = \begin{bmatrix} P(\lambda) & 0 & \cdots & 0 \\ \gamma P^{(1)}(\lambda) & P(\lambda) & \cdots & 0 \\ \frac{\gamma^2}{2!} P^{(2)}(\lambda) & \gamma P^{(1)}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma^{k-1}}{(k-1)!} P^{(k-1)}(\lambda) & \frac{\gamma^{k-2}}{(k-2)!} P^{(k-2)}(\lambda) & \cdots & P(\lambda) \end{bmatrix}, \quad k = 1, 2, \dots,$$

where $P^{(i)}(\lambda)$ denotes the i th derivative of $P(\lambda)$ with respect to λ . Then, he derived lower and upper bounds for a distance from $P(\lambda)$ to the matrix polynomials with a prescribed eigenvalue of a desired algebraic multiplicity, by generalizing the methodology used in [16]. Recently, Kokabifar, Loghmani, Nazari and Karbassi [9] extended the results of [16] to the case of two distinct eigenvalues, by replacing the first order derivative of $P(\lambda)$ in $F_2 [P(\lambda); \gamma]$ by a divided difference. Also, Karow and Mengi [8] studied systematically an alternative distance from a given $n \times n$ matrix polynomial to matrix polynomials with a specified number of eigenvalues at specified locations in the complex plane, deriving singular value optimization characterizations based on a Sylvester's equation characterization.

In this paper, motivated by the above spectrum updating problems, we introduce and study a (weighted) spectral norm distance from an $n \times n$ matrix polynomial $P(\lambda)$ to the set of all matrix polynomials with $k \leq n$ prescribed distinct eigenvalues. In particular, we obtain an upper and a lower bounds for this distance, and construct an optimal perturbation associated to the upper bound. Replacing the derivatives of $P(\lambda)$ in $F_k [P(\lambda); \gamma]$ by divided differences formulas, extending necessary definitions and lemmas of [9, 11, 16, 17], and constructing an appropriate perturbation of $P(\lambda)$ are the main ideas used herein. (Hence, this article can be considered as a generalization of the results obtained in [11] to the case of matrix polynomials, and also as an extension of [9, 16, 17] to the case of k arbitrary distinct eigenvalues). In the next section, we review standard definitions on matrix polynomials, and we also introduce some definitions which are necessary for the remainder. In Section 3, we construct an admissible perturbation of $P(\lambda)$ by extending the methods described in [9, 16, 17]. In Section 4, we obtain our bounds, and in Section 5, we give two numerical examples to illustrate the effectiveness of the proposed technique.

2 Preliminaries

In the last decades, the study of matrix polynomials, especially with regard to their spectral analysis, has received much attention of several researchers and has met many applications. Some basic references for the theory and applications of matrix polynomials are [4, 7, 10, 14, 19] and references therein.

For $A_j \in \mathbb{C}^{n \times n}$ ($j = 0, 1, \dots, m$) and a complex variable λ , we define the *matrix polynomial*

$$P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \dots + A_1 \lambda + A_0 = \sum_{j=0}^m A_j \lambda^j. \quad (1)$$

If for a scalar $\mu \in \mathbb{C}$ and some nonzero vector $v \in \mathbb{C}^n$, it holds that $P(\mu)v = 0$, then the scalar μ is called an *eigenvalue* of $P(\lambda)$ and the vector v is known as a (*right*) *eigenvector* of $P(\lambda)$ corresponding to μ . Similarly, a nonzero vector $\nu \in \mathbb{C}^n$ is known as a (*left*) *eigenvector* of $P(\lambda)$ corresponding to μ when $\nu^* P(\mu) = 0$. The *spectrum* of $P(\lambda)$, denoted by $\sigma(P)$, is the set of its eigenvalues. Throughout of this paper, it is assumed that the coefficient matrix A_m is *nonsingular*; this implies that the spectrum of $P(\lambda)$ contains no more than mn distinct elements.

The multiplicity of an eigenvalue $\lambda_0 \in \sigma(P)$ as a root of the scalar polynomial $\det P(\lambda)$ is called the *algebraic multiplicity* of λ_0 , and the dimension of the null space of the (constant) matrix $P(\lambda_0)$ is known as the *geometric multiplicity* of λ_0 . The algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity. An eigenvalue is called *semisimple* if its algebraic and geometric multiplicities are equal; otherwise, it is known as *defective*. The singular values of $P(\lambda)$ are the nonnegative roots of the eigenvalue functions of $P(\lambda)^* P(\lambda)$, and they are denoted by $s_1(P(\lambda)) \geq s_2(P(\lambda)) \geq \dots \geq s_n(P(\lambda))$ (i.e., they are considered in a nondecreasing order).

Definition 2.1. Let $P(\lambda)$ be a matrix polynomial as in (1) and let $\Delta_j \in \mathbb{C}^{n \times n}$ ($j = 0, 1, \dots, m$) be arbitrary matrices. Consider perturbations of the matrix polynomial $P(\lambda)$ of the form

$$Q(\lambda) = P(\lambda) + \Delta(\lambda) = \sum_{j=0}^m (A_j + \Delta_j) \lambda^j. \quad (2)$$

Also, for $\varepsilon > 0$ and a set of given nonnegative weights $w = \{w_0, w_1, \dots, w_m\}$, with $w_0 > 0$, define the class of admissible perturbed matrix polynomials

$$\mathcal{B}(P, \varepsilon, w) = \{Q(\lambda) \text{ as in (2)} : \|\Delta_j\|_2 \leq \varepsilon w_j, j = 0, 1, \dots, m\},$$

and the scalar polynomial $w(\lambda) = w_m \lambda^m + w_{m-1} \lambda^{m-1} + \dots + w_1 \lambda + w_0$.

Definition 2.2. Let $P(\lambda)$ be a matrix polynomial as in (1), and let a set of distinct complex numbers $\Sigma = \{\mu_1, \mu_2, \dots, \mu_k\}$ ($k \leq n$) be given. The distance from $P(\lambda)$ to the set of matrix polynomials whose spectra include Σ is defined and denoted by

$$D_w(P, \Sigma) = \min \{ \varepsilon \geq 0 : \exists Q(\lambda) \in \mathcal{B}(P, \varepsilon, w) \text{ such that } \Sigma \subseteq \sigma(Q) \}.$$

Definition 2.3. Consider a complex function f and k distinct scalars $\mu_1, \mu_2, \dots, \mu_k \in \mathbb{C}$. The *divided difference relative to μ_i and μ_{i+t}* ($1 \leq i \leq k-1$, $1 \leq t \leq k-i$) is denoted by $f[\mu_i, \mu_{i+1}, \dots, \mu_{i+t}]$ and is defined by the following recursive formula [3]:

$$f[\mu_i, \mu_{i+1}, \dots, \mu_{i+t}] = \frac{f[\mu_i, \mu_{i+1}, \dots, \mu_{i+t-1}] - f[\mu_{i+1}, \mu_{i+2}, \dots, \mu_{i+t}]}{\mu_i - \mu_{i+t}},$$

where $f[\mu_i] = f(\mu_i)$ ($i = 1, 2, \dots, k$).

Definition 2.4. Suppose that $P(\lambda)$ is a matrix polynomial as in (1) and a set of distinct complex numbers $\Sigma = \{\mu_1, \mu_2, \dots, \mu_k\}$ ($k \leq n$) is given. For any scalar $\gamma \in \mathbb{C}$, define the $nk \times nk$ matrix

$$F_\gamma[P, \Sigma] = \begin{bmatrix} P(\mu_1) & 0 & \cdots & 0 \\ \gamma P[\mu_1, \mu_2] & P(\mu_2) & \cdots & 0 \\ \gamma^2 P[\mu_1, \mu_2, \mu_3] & \gamma P[\mu_2, \mu_3] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{k-1} P[\mu_1, \dots, \mu_k] & \gamma^{k-2} P[\mu_2, \dots, \mu_k] & \cdots & P(\mu_k) \end{bmatrix}.$$

3 Construction of a perturbation

In this section, we construct an $n \times n$ matrix polynomial $\Delta_\gamma(\lambda)$ such that the given set of distinct scalars $\Sigma = \{\mu_1, \mu_2, \dots, \mu_k\}$ ($k \leq n$) is included in the spectrum of the perturbed matrix polynomial $Q_\gamma(\lambda) = P(\lambda) + \Delta(\lambda)$. Without loss of generality, hereafter we can assume that the parameter γ is real nonnegative [17]. Moreover, for convenience, we set $\rho = nk - k + 1$.

Definition 3.1. Suppose that

$$u(\gamma) = \begin{bmatrix} u_1(\gamma) \\ u_2(\gamma) \\ \vdots \\ u_k(\gamma) \end{bmatrix}, \quad v(\gamma) = \begin{bmatrix} v_1(\gamma) \\ v_2(\gamma) \\ \vdots \\ v_k(\gamma) \end{bmatrix} \in \mathbb{C}^{nk} \quad (u_j(\gamma), v_j(\gamma) \in \mathbb{C}^n, j = 1, 2, \dots, k)$$

is a pair of left and right singular vectors of $s_\rho(F_\gamma[P, \Sigma])$, respectively. Define the $n \times k$ matrices

$$U(\gamma) = [u_1(\gamma) \ u_2(\gamma) \ \cdots \ u_k(\gamma)] \quad \text{and} \quad V(\gamma) = [v_1(\gamma) \ v_2(\gamma) \ \cdots \ v_k(\gamma)].$$

Suppose now that $\gamma > 0$ and $\text{rank}(V(\gamma)) = k$. Define the quantities

$$\theta_{i,j} = \frac{\gamma}{\mu_i - \mu_j}, \quad i, j \in \{1, 2, \dots, k\}, \quad i \neq j, \quad (3)$$

and the vectors

$$\hat{v}_1(\gamma) = v_1(\gamma), \quad \hat{v}_p(\gamma) = v_p(\gamma) + \sum_{i=1}^{p-1} \left[(-1)^i \left(\prod_{j=p-i}^{p-1} \theta_{j,p} \right) v_{p-i}(\gamma) \right] \quad (p = 2, 3, \dots, k)$$

and

$$\hat{u}_1(\gamma) = u_1(\gamma), \quad \hat{u}_p(\gamma) = u_p(\gamma) + \sum_{i=1}^{p-1} \left[(-1)^i \left(\prod_{j=p-i}^{p-1} \theta_{j,p} \right) u_{p-i}(\gamma) \right] \quad (p = 2, 3, \dots, k).$$

Analogously to Definition 3.1, we define the $n \times k$ matrices

$$\hat{U}(\gamma) = [\hat{u}_1(\gamma) \ \hat{u}_2(\gamma) \ \cdots \ \hat{u}_k(\gamma)] \quad \text{and} \quad \hat{V}(\gamma) = [\hat{v}_1(\gamma) \ \hat{v}_2(\gamma) \ \cdots \ \hat{v}_k(\gamma)].$$

We also consider the quantities

$$\alpha_{i,s} = \frac{1}{w(|\mu_i|)} \sum_{j=0}^m \left(\left(\frac{\bar{\mu}_i}{|\mu_i|} \right)^j \mu_s^j w_j \right) \quad \text{and} \quad \beta_s = \frac{1}{k} \sum_{i=1}^k \alpha_{i,s}, \quad i, s = 1, 2, \dots, k, \quad (4)$$

where $w_0 > 0$ and, by convention, we set $\alpha_{i,s} = 1$ whenever $\mu_i = 0$. If $\beta_1, \beta_2, \dots, \beta_k$ are nonzero, then we define the $n \times n$ matrix

$$\Delta_\gamma = -s_\rho(F_\gamma[P, \Sigma]) \hat{U}(\gamma) \text{diag} \left\{ \frac{1}{\beta_1}, \frac{1}{\beta_2}, \dots, \frac{1}{\beta_k} \right\} \hat{V}(\gamma)^\dagger,$$

where $\hat{V}(\gamma)^\dagger$ denotes the *Moore-Penrose pseudoinverse* of $\hat{V}(\gamma)$, and the $n \times n$ matrix polynomial

$$\Delta_\gamma(\lambda) = \sum_{j=0}^m \Delta_{\gamma,j} \lambda^j,$$

where

$$\Delta_{\gamma,j} = \frac{1}{k} \sum_{i=1}^k \left(\frac{1}{w(|\mu_i|)} \left(\frac{\bar{\mu}_i}{|\mu_i|} \right)^j w_j \right) \Delta_\gamma, \quad j = 0, 1, \dots, m. \quad (5)$$

By straightforward computations, we verify that the matrix polynomial $\Delta_\gamma(\lambda)$ satisfies

$$\Delta_\gamma(\mu_s) = \sum_{j=0}^m \left[\frac{1}{k} \sum_{i=1}^k \left(\frac{1}{w(|\mu_i|)} \left(\frac{\bar{\mu}_i}{|\mu_i|} \right)^j \right) w_j \mu_s^j \right] \Delta_\gamma = \beta_s \Delta_\gamma, \quad s = 1, 2, \dots, k.$$

Notice that the condition $\text{rank}(V(\gamma)) = k$ implies $\hat{v}_i(\gamma) \neq 0$, ($i = 1, 2, \dots, k$) and $\hat{V}(\gamma)^\dagger \hat{V}(\gamma) = I_k$, where I_k denotes the $k \times k$ identity matrix.

Moreover, since $u(\gamma), v(\gamma)$ is a pair of left and right singular vectors of $s_\rho(F_\gamma[P, \Sigma])$, we have

$$F_\gamma[P, \Sigma]v(\gamma) = s_\rho(F_\gamma[P, \Sigma])u(\gamma),$$

or equivalently, the following hold:

$$\begin{aligned} s_\rho(F_\gamma[P, \Sigma])u_1(\gamma) &= P(\mu_1)v_1(\gamma), \\ s_\rho(F_\gamma[P, \Sigma])u_2(\gamma) &= \gamma P[\mu_1, \mu_2]v_1(\gamma) + P(\mu_2)v_2(\gamma), \\ &\vdots \\ s_\rho(F_\gamma[P, \Sigma])u_k(\gamma) &= \gamma^{k-1}P[\mu_1, \dots, \mu_k]v_1(\gamma) + \gamma^{k-2}P[\mu_2, \dots, \mu_k]v_2(\gamma) + \dots + P(\mu_k)v_k(\gamma). \end{aligned}$$

Substituting $\hat{u}_1(\gamma), \hat{u}_2(\gamma), \dots, \hat{u}_k(\gamma)$ and $\hat{v}_1(\gamma), \hat{v}_2(\gamma), \dots, \hat{v}_k(\gamma)$ into these equations yields

$$s_\rho(F_\gamma[P, \Sigma])\hat{u}_i(\gamma) = P(\mu_i)\hat{v}_i(\gamma), \quad i = 1, 2, \dots, k.$$

Therefore, for the matrix polynomial

$$Q_\gamma(\lambda) = P(\lambda) + \Delta_\gamma(\lambda) = \sum_{j=0}^m (A_j + \Delta_{\gamma,j})\lambda^j \quad (6)$$

(recall the coefficient perturbations $\Delta_{\gamma,j}$ in (5)), and for every $i = 1, 2, \dots, k$, it follows

$$\begin{aligned} Q_\gamma(\mu_i)\hat{v}_i(\gamma) &= P(\mu_i)\hat{v}_i(\gamma) + \Delta_\gamma(\mu_i)\hat{v}_i(\gamma) \\ &= s_\rho(F_\gamma[P, \Sigma])\hat{u}_i(\gamma) + \beta_i\Delta_\gamma\hat{v}_i(\gamma) \\ &= s_\rho(F_\gamma[P, \Sigma])\hat{u}_i(\gamma) + \beta_i\left(-s_\rho(F_\gamma[P, \Sigma])\frac{1}{\beta_i}\right)\hat{u}_i(\gamma) \\ &= 0. \end{aligned}$$

As a consequence, if $\text{rank}(V(\gamma)) = k$ (recall that all $\beta_1, \beta_2, \dots, \beta_k$ in (4) are nonzero), then $\mu_1, \mu_2, \dots, \mu_k$ are eigenvalues of the matrix polynomial $Q_\gamma(\lambda)$ in (6) with $\hat{v}_1(\gamma), \hat{v}_2(\gamma), \dots, \hat{v}_k(\gamma)$ as their associated eigenvectors, respectively.

The next result follows immediately.

Theorem 3.2. *Consider a matrix polynomial $P(\lambda)$ as in (1) and a given set of $k \leq n$ distinct complex numbers $\Sigma = \{\mu_1, \mu_2, \dots, \mu_k\}$, and suppose that the quantities $\beta_1, \beta_2, \dots, \beta_k$ in (4) are nonzero. For every $\gamma > 0$ such that $\text{rank}(V(\gamma)) = k$, the scalars $\mu_1, \mu_2, \dots, \mu_k$ are eigenvalues of the matrix polynomial $Q_\gamma(\lambda)$ in (6), with corresponding eigenvectors $\hat{v}_1(\gamma), \hat{v}_2(\gamma), \dots, \hat{v}_k(\gamma)$, respectively.*

Remark 3.3. For the case $k = 2$, by [9, Section 2] (see also [16, Section 5]), if the matrix $P[\mu_1, \mu_2]$ is nonsingular and $\gamma_* > 0$ is a point where the singular value $s_{2n-1}(F_\gamma[P, \{\mu_1, \mu_2\}])$ attains its maximum value, then $\text{rank}(V(\gamma_*)) = 2 (= k)$. But for the case $k > 2$, as mentioned in [17], it is not easy to obtain conditions ensuring $\text{rank}(V(\gamma)) = k$. However, in all our experiments, the condition $\text{rank}(V(\gamma)) = k$ holds generically. Also, $\beta_1, \beta_2, \dots, \beta_k \neq 0$ appears to be generic.

4 Bounds for $D_w(P, \Sigma)$

The construction of the perturbed matrix polynomial $Q_\gamma(\lambda)$ in (6) yields immediately an upper bound for the distance $D_w(P, \Sigma)$. In particular, from (5) we have

$$\|\Delta_{\gamma,j}\|_2 \leq \frac{w_j}{k} \sum_{i=1}^k \left(\frac{1}{w(|\mu_i|)} \right) \|\Delta_\gamma\|_2, \quad j = 0, 1, \dots, k.$$

Consequently, if all $\beta_1, \beta_2, \dots, \beta_k$ in (4) are nonzero, then for any $\gamma > 0$ such that $\text{rank}(V(\gamma)) = k$, it follows

$$D_w(P, \Sigma) \leq \frac{1}{k} \sum_{i=1}^k \left(\frac{1}{w(|\mu_i|)} \right) \|\Delta_\gamma\|_2. \quad (7)$$

Next, we compute a lower bound for $D_w(P, \Sigma)$. It is worth mentioning that for calculating this lower bound, the condition $\text{rank}(V(\gamma)) = k$ is not necessary.

Lemma 4.1. *Suppose that $P(\lambda)$ is a matrix polynomial as in (1), and $\mu_1, \mu_2, \dots, \mu_k$ are k distinct eigenvalues of $P(\lambda)$. Then, for every $\gamma > 0$, it holds that $s_\rho(F_\gamma[P, \Sigma]) = 0$ (recall that $\rho = nk - k + 1$).*

Proof. Since $\mu_1, \mu_2, \dots, \mu_k$ are distinct eigenvalues of $P(\lambda)$, there exist k nonzero (but not necessarily linearly independent) vectors $\nu_1, \nu_2, \dots, \nu_k$ satisfying $P(\mu_i)\nu_i = 0$, $i = 1, 2, \dots, k$.

Recalling Definition 2.4 and the quantities $\theta_{i,j}$ ($i \neq j$) defined by (3), the $nk \times nk$ matrix $F_\gamma[P, \Sigma]$ can be written in the form

$$F_\gamma[P, \Sigma] = \begin{bmatrix} P(\mu_1) & 0 & 0 & \cdots & 0 \\ \theta_{1,2}(P(\mu_1) - P(\mu_2)) & P(\mu_2) & 0 & \cdots & 0 \\ \theta_{1,3}[\theta_{1,2}P(\mu_1) - (\theta_{1,2} + \theta_{2,3})P(\mu_2) + \theta_{2,3}P(\mu_3)] & \theta_{2,3}(P(\mu_2) - P(\mu_3)) & P(\mu_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & P(\mu_k) \end{bmatrix}.$$

Denoting the (i, j) th $n \times n$ block of this matrix by $F_{i,j}$, it follows readily that

$$F_{i,j} = \theta_{j,i}(F_{i-1,j} + F_{i,j+1}), \quad 1 \leq j < i \leq k. \quad (8)$$

Moreover, for all distinct i, j and q in $\{1, 2, \dots, k\}$, it holds that

$$\theta_{i,j}(\theta_{i,q} + \theta_{q,j}) = \frac{\gamma}{\mu_i - \mu_j} \frac{\gamma(\mu_i - \mu_j)}{(\mu_i - \mu_q)(\mu_q - \mu_j)} = \theta_{i,q} \theta_{q,j}. \quad (9)$$

By straightforward calculations, and using (8) and (9), one can verify that the k (nonzero) linearly independent vectors

$$\begin{bmatrix} \nu_1 \\ \theta_{1,2}\nu_1 \\ \theta_{1,2}\theta_{1,3}\nu_1 \\ \vdots \\ \left(\prod_{j=2}^{k-1} \theta_{1,j}\right) \nu_1 \\ \left(\prod_{j=2}^k \theta_{1,j}\right) \nu_1 \end{bmatrix}, \begin{bmatrix} 0 \\ \nu_2 \\ \theta_{2,3}\nu_2 \\ \vdots \\ \left(\prod_{j=3}^{k-1} \theta_{2,j}\right) \nu_2 \\ \left(\prod_{j=3}^k \theta_{2,j}\right) \nu_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \nu_3 \\ \vdots \\ \left(\prod_{j=4}^{k-1} \theta_{3,j}\right) \nu_3 \\ \left(\prod_{j=4}^k \theta_{3,j}\right) \nu_3 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \nu_{k-1} \\ \theta_{k-1,k}\nu_{k-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \nu_k \end{bmatrix}$$

lie in the null space of the matrix $F_\gamma[P, \Sigma]$. Thus, the rank of $F_\gamma[P, \Sigma]$ is less than or equal to $kn - k = \rho - 1$, and the proof is complete. \square

The next lemma yields a lower bound of $D_w(P, \Sigma)$. We need to define the nonnegative quantities

$$\varpi[\mu_i] = w(|\mu_i|), \quad i = 1, 2, \dots, k,$$

$$\varpi[\mu_i, \mu_{i+1}] = \sum_{j=0}^m w_j \frac{|\mu_i^j - \mu_{i+1}^j|}{|\mu_i - \mu_{i+1}|}, \quad i = 1, 2, \dots, k-1,$$

and (recursively)

$$\varpi[\mu_i, \mu_{i+1}, \dots, \mu_{i+t}] = \frac{\varpi[\mu_i, \dots, \mu_{i+t-1}] + \varpi[\mu_{i+1}, \dots, \mu_{i+t}]}{|\mu_i - \mu_{i+t}|}, \quad i = 1, 2, \dots, k-2, \quad t = 2, 3, \dots, k-i,$$

and the $k \times k$ matrix

$$F_\gamma[\varpi, \Sigma] = \begin{bmatrix} \varpi[\mu_1] & 0 & \cdots & 0 \\ \gamma\varpi[\mu_1, \mu_2] & \varpi[\mu_2] & \cdots & 0 \\ \gamma^2\varpi[\mu_1, \mu_2, \mu_3] & \gamma\varpi[\mu_2, \mu_3] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{k-1}\varpi[\mu_1, \mu_2, \dots, \mu_k] & \gamma^{k-2}\varpi[\mu_2, \mu_3, \dots, \mu_k] & \cdots & \varpi[\mu_k] \end{bmatrix}.$$

Lemma 4.2. *Suppose that the matrix polynomial $Q(\lambda) = P(\lambda) + \Delta(\lambda)$ belongs to $\mathcal{B}(P, \varepsilon, w)$. If k distinct scalars $\mu_1, \mu_2, \dots, \mu_k \in \mathbb{C}$ are eigenvalues of $Q(\lambda)$, then for any $\gamma > 0$,*

$$\varepsilon \geq \frac{s_\rho(F_\gamma[P, \Sigma])}{\|F_\gamma[\varpi, \Sigma]\|_2}. \quad (10)$$

Proof. It is easy to see that

$$\begin{aligned} \|\Delta(\mu_i)\|_2 &\leq \sum_{j=0}^m \|\Delta_j\|_2 |\mu_i|^j \leq \varepsilon \sum_{j=0}^m w_j |\mu_i|^j = \varepsilon w(|\mu_i|) = \varepsilon \varpi[\mu_i], \quad i = 1, 2, \dots, k, \\ \|\Delta[\mu_i, \mu_{i+1}]\|_2 &\leq \sum_{j=0}^m \|\Delta_j\|_2 \left| \frac{\mu_i^j - \mu_{i+1}^j}{\mu_i - \mu_{i+1}} \right| \leq \varepsilon \varpi[\mu_i, \mu_{i+1}], \quad i = 1, 2, \dots, k-1, \end{aligned}$$

and

$$\begin{aligned} \|\Delta[\mu_i, \mu_{i+1}, \mu_{i+2}]\|_2 &\leq \frac{1}{|\mu_i - \mu_{i+2}|} (\|\Delta[\mu_i, \mu_{i+1}]\|_2 + \|\Delta[\mu_{i+1}, \mu_{i+2}]\|_2) \\ &\leq \frac{1}{|\mu_i - \mu_{i+2}|} \left(\varepsilon \sum_{j=0}^m w_j \frac{|\mu_i^j - \mu_{i+1}^j|}{|\mu_i - \mu_{i+1}|} + \varepsilon \sum_{j=0}^m w_j \frac{|\mu_{i+1}^j - \mu_{i+2}^j|}{|\mu_{i+1} - \mu_{i+2}|} \right) \\ &\leq \varepsilon \varpi[\mu_i, \mu_{i+1}, \mu_{i+2}], \quad i = 1, 2, \dots, k-2. \end{aligned}$$

Similarly, we can obtain

$$\|\Delta[\mu_i, \dots, \mu_{i+t}]\|_2 \leq \varepsilon \varpi[\mu_i, \dots, \mu_{i+t}], \quad i = 1, 2, \dots, k-2, \quad t = 2, 3, \dots, k-i.$$

As in the proof of Theorem 2.4 of [17], we can consider a unit vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{C}^{kn} \quad (x_i \in \mathbb{C}^n, \quad i = 1, 2, \dots, k)$$

such that

$$\begin{aligned}
\|F_\gamma [\Delta, \Sigma]\|_2^2 &= \|F_\gamma [\Delta, \Sigma] x\|_2^2 \\
&= \|\Delta (\mu_1) x_1\|_2^2 + \|\gamma \Delta [\mu_1, \mu_2] x_1 + \Delta (\mu_2) x_2\|_2^2 \\
&\quad + \cdots + \left\| \sum_{i=1}^k \gamma^{k-i} \Delta [\mu_i, \dots, \mu_k] x_i \right\|_2^2 \\
&\leq (\varepsilon \varpi [\mu_1])^2 \|x_1\|_2^2 + (\gamma \varepsilon \varpi [\mu_1, \mu_2])^2 \|x_1\|_2^2 + (\varepsilon \varpi [\mu_2])^2 \|x_2\|_2^2 \\
&\quad + 2\gamma (\varepsilon \varpi [\mu_1, \mu_2]) (\varepsilon \varpi [\mu_2]) \|x_1\|_2 \|x_2\|_2 + \cdots + (\varepsilon \varpi [\mu_k])^2 \|x_k\|_2^2 \\
&= \varepsilon^2 \left\| \begin{bmatrix} \varpi [\mu_1] & 0 & \cdots & 0 \\ \gamma \varpi [\mu_1, \mu_2] & \varpi [\mu_2] & \cdots & 0 \\ \gamma^2 \varpi [\mu_1, \mu_2, \mu_3] & \gamma \varpi [\mu_2, \mu_3] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{k-1} \varpi [\mu_1, \mu_2, \dots, \mu_k] & \gamma^{k-2} \varpi [\mu_2, \mu_3, \dots, \mu_k] & \cdots & \varpi [\mu_k] \end{bmatrix} \begin{bmatrix} \|x_1\|_2 \\ \|x_2\|_2 \\ \vdots \\ \|x_k\|_2 \end{bmatrix} \right\|_2^2 \\
&\leq \varepsilon^2 \|F_\gamma [\varpi, \Sigma]\|_2^2.
\end{aligned}$$

Moreover, since the k distinct scalars $\mu_1, \mu_2, \dots, \mu_k$ are eigenvalues of $Q(\lambda) = P(\lambda) + \Delta(\lambda)$, Lemma 4.1 implies that $s_\rho(F_\gamma [Q, \Sigma]) = 0$. Applying the Weyl inequalities (e.g., see Corollary 5.1 of [1]) for singular values, with respect to the relation $F_\gamma [Q, \Sigma] = F_\gamma [P, \Sigma] + F_\gamma [\Delta, \Sigma]$, yields

$$s_\rho(F_\gamma [P, \Sigma]) \leq \|F_\gamma [\Delta, \Sigma]\|_2 \leq \varepsilon \|F_\gamma [\varpi, \Sigma]\|_2$$

for any $\gamma > 0$. This completes the proof. \square

Keeping in mind Definition 2.2, the above lemma yields a lower bound for $D_w(P, \Sigma)$, namely,

$$D_w(P, \Sigma) \geq \frac{s_\rho(F_\gamma [P, \Sigma])}{\|F_\gamma [\varpi, \Sigma]\|_2}. \quad (11)$$

It will be convenient to denote the lower bound in (11) by $\beta_{low}(P, \Sigma, \gamma)$ and the upper bound in (7) by $\beta_{up}(P, \Sigma, \gamma)$, i.e.,

$$\beta_{low}(P, \Sigma, \gamma) = \frac{s_\rho(F_\gamma [P, \Sigma])}{\|F_\gamma [\varpi, \Sigma]\|_2}, \quad (12)$$

and

$$\beta_{up}(P, \Sigma, \gamma) = \frac{1}{k} \sum_{i=1}^k \left(\frac{1}{w(|\mu_i|)} \right) \|\Delta_\gamma\|_2, \quad (13)$$

Our results so far are summarized in the following theorem.

Theorem 4.3. Consider an $n \times n$ matrix polynomial $P(\lambda)$ as in (1) and a given set of $k \leq n$ distinct complex numbers $\Sigma = \{\mu_1, \mu_2, \dots, \mu_k\}$.

- (a) For any $\gamma > 0$, $D_w(P, \Sigma) \geq \beta_{low}(P, \Sigma, \gamma)$.
- (b) If the quantities $\beta_1, \beta_2, \dots, \beta_k$ in (4) are nonzero, then for any $\gamma > 0$ such that $\text{rank}(V(\gamma)) = k$, $D_w(P, \Sigma) \leq \beta_{up}(P, \Sigma, \gamma)$ and the matrix polynomial $Q_\gamma(\gamma)$ in (6) lies on the boundary of $\mathcal{B}(P, \beta_{up}(P, \Sigma, \gamma), w)$.

Next we consider the case $\gamma = 0$. For $i = 1, 2, \dots, k$, let $\tilde{u}_i, \tilde{v}_i \in \mathbb{C}^n$ be a pair of left and right singular vectors of $P(\mu_i)$ corresponding to $\sigma_i = s_n(P(\mu_i))$, respectively. If the vectors $\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_k$ are linearly independent, then we define the constant matrix

$$\Delta_0 = -[\tilde{u}_1 \ \tilde{u}_1 \ \cdots \ \tilde{u}_k] \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_k\} [\tilde{v}_1 \ \tilde{v}_2 \ \cdots \ \tilde{v}_k]^\dagger$$

and observe that $[\tilde{v}_1 \ \tilde{v}_2 \ \cdots \ \tilde{v}_k]^\dagger [\tilde{v}_1 \ \tilde{v}_2 \ \cdots \ \tilde{v}_k] = I_k$. Therefore, the matrix polynomial

$$Q_0(\lambda) = P(\lambda) + \Delta_0(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + (A_0 + \Delta_0), \quad (14)$$

lies on the boundary of $\mathcal{B}\left(P, \frac{\|\Delta_0\|_2}{w_0}, w\right)$ and satisfies

$$Q_0(\mu_i) \tilde{v}_i = P(\mu_i) \tilde{v}_i + \Delta_0(\mu_i) \tilde{v}_i = \sigma_i \tilde{u}_i - \sigma_i \tilde{u}_i = 0, \quad i = 1, 2, \dots, k.$$

Hence, the scalars $\mu_1, \mu_2, \dots, \mu_k$ are eigenvalues of the matrix polynomial $Q_0(\lambda)$ in (14) with corresponding eigenvectors $\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_k$, respectively.

Theorem 4.4. Let $\gamma = 0$, and let $\tilde{u}_i, \tilde{v}_i \in \mathbb{C}^n$ be a pair of left and right singular vectors of $P(\mu_i)$ corresponding to $\sigma_i = s_n(P(\mu_i))$, respectively, for every $i = 1, 2, \dots, k$. If the vectors $\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_k$ are linearly independent, then the matrix polynomial $Q_0(\lambda)$ in (14) lies on the boundary of $\mathcal{B}\left(P, \frac{\|\Delta_0\|_2}{w_0}, w\right)$ and has $\mu_1, \mu_2, \dots, \mu_k$ as eigenvalues.

In the next remark, we give an upper and a lower bounds for a spectral norm distance from an $n \times n$ matrix A to the set of all matrices with k prescribed eigenvalues. This issue is explained in [11] in detail.

Remark 4.5. Consider the standard eigenproblem of a matrix $A \in \mathbb{C}^{n \times n}$. In this special case, we set $P(\lambda) = I\lambda - A$ and $w = \{w_0, w_1\} = \{1, 0\}$. Thus, for every $i = 1, 2, \dots, k$, $\varpi[\mu_i] = w(|\mu_i|) = w_0$ and $\varpi[\mu_i, \dots, \mu_j] = 0$ for every $j = \{i+1, i+2, \dots, k\}$. Consequently, the matrix $F_\gamma[\varpi, \Sigma]$ becomes the identity matrix I_k and the lower bound in (12) turns into $\beta_{low}(P, \Sigma, \gamma) = s_\rho(F_\gamma[P, \Sigma])$. Furthermore, it is easy to see that $\alpha_{i,s} = 1$ and $\beta_s = 1$ for every $i, s = 1, 2, \dots, k$. Therefore, the upper bound in (13) becomes

$$\beta_{up}(P, \Sigma, \gamma) = \|\Delta_\gamma\|_2 = s_\rho(F_\gamma[P, \Sigma]) \left\| \hat{U}(\gamma) \hat{V}(\gamma)^\dagger \right\|_2.$$

Moreover, the associated perturbed matrix polynomial $Q_\gamma(\lambda)$ in (6) is now written

$$Q_\gamma(\lambda) = P(\lambda) + \Delta_\gamma(\lambda) = P(\lambda) + \Delta_\gamma = I\lambda - \left(A + s_\rho(F_\gamma[P, \Sigma]) \hat{U}(\gamma) \hat{V}(\gamma)^\dagger \right). \quad (15)$$

5 Numerical examples

In this section, the validity of the method described in the previous sections is verified by two numerical examples. The lower and upper bounds for the distance $D_w(P, \Sigma)$ are computed by applying the procedures described in Section 4, and by using the MATLAB function `fminbnd` which finds a minimum of a function of one variable within a fixed interval. As it was mentioned in Remark 3.3, the condition $\text{rank}(V(\gamma)) = k$ appears to be generic when $\gamma > 0$. All computations were performed in MATLAB with 16 significant digits; however, for simplicity, all numerical results are shown with 4 decimal places.

Example 5.1. Consider the 3×3 matrix polynomial

$$P(\lambda) = \begin{bmatrix} 7 & 9 & -2 \\ 0 & -2 & 0 \\ 6 & -3 & -1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 9 & -3 & 3 \\ -5 & 8 & 10 \\ 4 & -3 & 0 \end{bmatrix} \lambda + \begin{bmatrix} -5 & 0 & 5 \\ -2 & -2 & 10 \\ 1 & 9 & 2 \end{bmatrix},$$

whose spectrum is $\sigma(P) = \{76.9807, 0.9284, 0.3034, -1.0283, -0.9421 \pm 0.9281 i\}$. Let $w = \{w_0, w_1, w_2\} = \{12.0731, 14.8523, 11.7991\}$ be the set of weights which are the norms of the coefficient matrices, and suppose that the set of desired eigenvalues is $\Sigma = \{1 + i, -2, 3\}$. By applying the MATLAB function `fminbnd`, it appears that the function $\beta_{up}(P, \{1 + i, -2, 3\}, \gamma)$ ($\gamma > 0$) attains its minimum at $\gamma = 1.9656$, that is,

$$\beta_{up}(P, \{1 + i, -2, 3\}, 1.9656) = 1.0090,$$

and the function $\beta_{low}(P, \{1 + i, -2, 3\}, \gamma)$ ($\gamma > 0$) attains its maximum at $\gamma = 5.3634 \cdot 10^{-5}$, that is,

$$\beta_{low}(P, \{1 + i, -2, 3\}, 5.3634 \cdot 10^{-5}) = 0.1320.$$

In Figure 1, the graphs of the upper bound $\beta_{up}(P, \{1 + i, -2, 3\}, \gamma)$ and the lower bound $\beta_{low}(P, \{1 + i, -2, 3\}, \gamma)$ are plotted for $\gamma \in (0, 10]$. Also, for the perturbation

$$\begin{aligned} \Delta_{1.9656}(\lambda) = & \begin{bmatrix} -1.5506 + 0.5852 i & -3.6805 - 3.7560 i & 3.2843 - 2.4550 i \\ -1.3951 + 1.1287 i & 0.8130 - 3.6071 i & 1.4666 + 0.2551 i \\ -4.9524 + 1.3272 i & -0.1817 - 0.1712 i & -0.1517 - 2.5523 i \end{bmatrix} \lambda^2 \\ & + \begin{bmatrix} -1.0045 + 0.6941 i & -3.2991 - 2.0307 i & 1.9114 - 2.3391 i \\ -0.7966 + 1.0550 i & -0.0602 - 2.7233 i & 1.0938 - 0.0784 i \\ -3.3045 + 1.8295 i & -0.1603 - 0.0901 i & -0.5623 - 1.7977 i \end{bmatrix} \lambda \\ & + \begin{bmatrix} -2.1779 - 1.0042 i & 0.1345 - 7.6081 i & 5.8658 + 0.8927 i \\ -2.5802 - 0.2920 i & 4.5439 - 2.8248 i & 1.2263 + 1.7709 i \\ -6.3971 - 3.7574 i & -0.0080 - 0.3612 i & 2.4770 - 2.7481 i \end{bmatrix} \end{aligned}$$

the perturbed matrix polynomial $Q_{1.9656}(\lambda) = P(\lambda) + \Delta_{1.9656}(\lambda)$ lies on the boundary of the set $\mathcal{B}(P, \beta_{up}(P, \{1 + i, -2, 3\}, 1.9656), w) = \mathcal{B}(P, 1.0090, w)$ and has Σ in its spectrum.

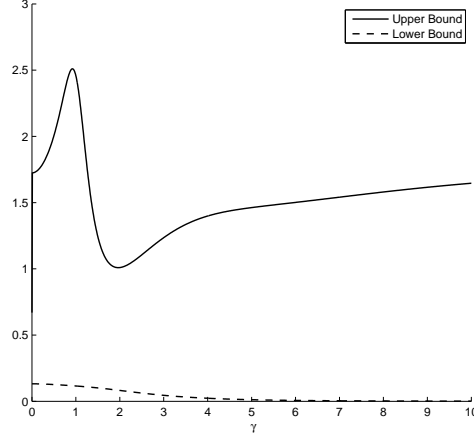


Fig 1: The graphs of $\beta_{low}(P, \{1 + i, -2, 3\}, \gamma)$ and $\beta_{up}(P, \{1 + i, -2, 3\}, \gamma)$.

Let us now consider the case $\gamma = 0$. Then, our discussion yields the perturbation

$$\Delta_0(\lambda) = \Delta_0 = \begin{bmatrix} 0.0673 + 0.0158i & 0.0656 - 0.0194i & 0.0060 - 0.0079i \\ 1.2669 - 0.1878i & 0.0412 + 0.2304i & -0.6315 + 0.0940i \\ 0.3092 - 0.1368i & -0.1210 + 0.1678i & -0.2397 + 0.0684i \end{bmatrix} \cdot 10^2,$$

and the perturbed matrix polynomial $Q_0(\lambda) = P(\lambda) + \Delta_0$ lies on the boundary of $\mathcal{B}(P, 12.5337, w)$ and has Σ in its spectrum.

Our second example illustrates the applicability of Remark 4.5.

Example 5.2. Consider the Frank matrix of order 12,

$$F_{12} = \begin{bmatrix} 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 11 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 10 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 9 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 8 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

which has some small ill-conditioned eigenvalues. Suppose that the set of the desired eigenvalues is $\Sigma = \{0.1, -0.1, 0.1i, -0.1i\}$. The optimal (spectral norm) distance from F_{12}

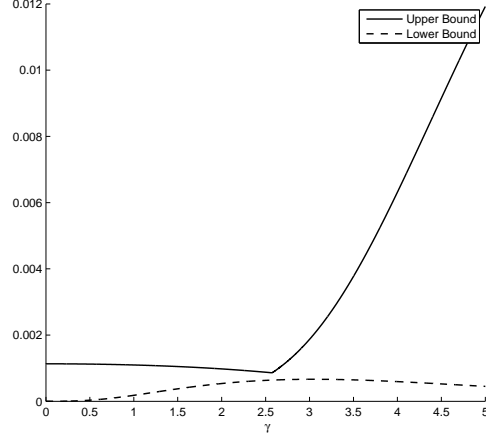


Fig 2: The graphs of $\beta_{low}(P, \Sigma, \gamma)$ and $\beta_{up}(P, \Sigma, \gamma)$.

to the set of matrices that have Σ in their spectrum is $6.9 \cdot 10^{-4}$ [11]. We consider the linear matrix polynomial $P(\lambda) = \lambda I_{12} - F_{12}$, and the weights $w_0 = 1$ and $w_1 = 0$ (i.e., we consider perturbations of the standard eigenproblem of matrix F_{12}). The MATLAB function `fminbnd` applied for the difference

$$\beta_{up}(P, \{0.1, -0.1, 0.1i, -0.1i\}, \gamma) - \beta_{low}(P, \{0.1, -0.1, 0.1i, -0.1i\}, \gamma)$$

yields $\gamma = 2.5730$. Then, according to the discussion in Remark 4.5, we have

$$\begin{aligned} \beta_{low}(P, \Sigma, 2.5730) = 6.4007 \cdot 10^{-4} &\leq 6.9 \cdot 10^{-4} = D_w(P, \Sigma) \\ &\leq 8.6167 \cdot 10^{-4} = \beta_{up}(P, \Sigma, 2.5730). \end{aligned}$$

Also, it is easy to see that the spectrum of the perturbed linear matrix polynomial $Q_\gamma(\lambda)$ in (15) includes the given set Σ . In Figure 2, the graphs of the upper bound $\beta_{up}(P, \Sigma, \gamma)$ and the lower bound $\beta_{low}(P, \Sigma, \gamma)$ are plotted for $\gamma \in (0, 5]$.

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